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Algebraic Models for Probability Measures Associated with Solutions of Random Equations, I

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1. INTRODUCTION

Let $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ be measurable spaces where X and Y are Banach spaces, and $\mathscr{B}$ and $\mathscr{C}$ are σ-algebras of Borel subsets of X and Y, respectively. In virtually all areas of applied mathematics we encounter operator equations of the form

$$Tx = y \tag{1}$$

where $T: X \to Y$ is a linear (or nonlinear) transformation. Now suppose T is a measurable transformation (cf. [7, p. 162]) and y is a Y-valued random element. Hence the operator equation in this case is of the form

$$Tx = y(\omega), \tag{2}$$

where $y: \Omega \to Y$, and $(\Omega, \mathfrak{A}, \mu)$ is a given probability measure space. Let ν_2 denote the probability measure induced on $(Y, \mathscr{C})$ by y and μ; hence ν_2 is the probability measure associated with the *random input* y. The *output* $x(\omega)$ is said to be a *random solution* of Eq. (2) if $Tx(\omega) = y(\omega)$ almost surely.

The following question is of fundamental importance in the theory of random equations: *Does there exist a probability measure* ν_1 *on* $(X, \mathscr{B})$ *such that*

$$\nu_1(T^{-1}(C)) = \nu_2(C) \tag{3}$$

for each $C \in \mathscr{C}$? In general there is not. This question has been considered recently by Eršov [4]. If such a measure ν_1 exists we call it the *output* (or *solution*) *measure* associated with x. The existence of solution measures, and the study of their properties, is of importance in the theory of random equations and its applications. We refer, in particular, to (1) the classical results of Cameron and Martin on transformations of Wiener measure on spaces of continuous functions

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(cf. [13, Chap. 8]), (2) solution measures associated with random differential and integral equations (Pun and Bharucha-Reid [8], Šatašvili [10, 11]), and (3) absolute continuity of output measures with respect to input measures (Bharucha-Reid [1, Chap. 1], Gikhman and Skorohod [5, 6, Chap. 7]).

In this paper we begin the study of the relationship between the probability measures ν_1 and ν_2 using the notion of an algebraic model for a measure space due to Dinculeanu and Foiaş [3]. In subsequent papers we will consider (1) the case when T is a random transformation, (2) transformations of Gaussan measures (using the results of [2]), and (3) the absolute continuity of output measures with respect to input measures.

2. Algebraic Models for Measure Spaces

In this section we state some basic definitions and results concerning algebraic models for measures that will be used in the next section of this paper, and in subsequent papers.

Definition 1. Let Γ be an Abelian group, and let φ be a function on Γ into the complex numbers Z with the following properties:

(i) φ is positive-definite; i.e., for every choice of elements $\gamma_1, \gamma_2, ..., \gamma_n$ in Γ and $z_1, z_2, ..., z_n$ in Z,

$$\sum_{i,j} z_i \bar{z}_j \varphi(\gamma_i \gamma_j^{-1}) \geqslant 0;$$

(ii) $\varphi(\gamma) = 1$ if and only if γ is the identity of Γ.

The pair (Γ, φ) is said to be an *algebraic measure system.*

Definition 2. Let $(\Omega, \mathfrak{A}, \mu)$ be a probability measure space. Let $H(\mu)$ denote the group of all complex-valued measurable functions on Ω to the unit circle. An algebraic measure system is said to be an *algebraic model for* $(\Omega, \mathfrak{A}, \mu)$, or simply μ, if there is an injective homomorphism $h: \Gamma \to H(\mu)$ such that

(i) the linear span of $h(\Gamma)$ is dense in $L_2(\mu)$, and

(ii) $\varphi(\gamma) = \int_\Omega h(\gamma)\, d\mu(\omega)$.

Definition 3. Two algebraic measure systems (Γ, φ) and $(\tilde{\Gamma}, \tilde{\varphi})$ are said to be *isomorphic* if there is a group isomorphism Ψ from Γ onto $\tilde{\Gamma}$ such that $\varphi(\gamma) = \tilde{\varphi}(\Psi\gamma)$

We now state the isomorphism theorem for algebraic models ([3], Theorem 2).

THEOREM 1. *Let $(\Omega, \mathfrak{A}, \mu)$ and $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mu})$ be two probability measure spaces. Then their measure algebras are isomorphic if and only if there exists a pair of isomorphic algebraic models.*

We remark that an equivalent, and perhaps more suitable, characterization of isomorphic measure algebras is as follows: Two probability measure spaces $(\Omega, \mathfrak{A}, \mu)$ and $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mu})$ have isomorphic measure algebras if and only if they are *conjugate*; that is, there is a linear isometry λ of $L_2(\mu)$ onto $L_2(\tilde{\mu})$ such that

(i) $\lambda L_\infty(\mu) = L_\infty(\tilde{\mu})$,

and

(ii) $\lambda(fg) = \lambda(f)\,\lambda(g)$ for all $f, g \in L_\infty(\mu)$.

The notion of algebraic models for probability measure spaces was introduced in the theory of stochastic processes by Schreiber, Sun, and Bharucha-Reid [12]. In particular, they utilized algebraic models to obtain Kolmogorov extension theorems for probability measures on abstract spaces.

3. ALGEBRAIC MODELS FOR TRANSFORMED MEASURES

In this section we consider the question posed in Section 1 using algebraic models. That is, if there is an algebraic model for $(Y, \mathscr{C}, \nu_2)$, we construct an algebraic measure system which, if ν_1 exists, will be an algebraic model for $(X, \mathscr{B}, \nu_1)$.

As before, let $T: X \to Y$ be a measurable transformation and let $(\Gamma(Y), \varphi_2)$ be an algebraic model for $(Y, \mathscr{C}, \nu_2)$. Since $(\Gamma(Y), \varphi_2)$ is an algebraic model for $(Y, \mathscr{C}, \nu_2)$ there exists an injective homomorphism $h: \Gamma(Y) \to H(\nu_2)$, the group of $\mathscr{C}$-measurable unimodular functions on Y, which satisfies

(i) $\varphi_2(\gamma) = \int_Y h(\gamma)\, d\nu$.

(ii) Span of $h(\Gamma(Y))$ is dense in $L_2(\nu_2)$.

Therefore, for each $\gamma \in \Gamma(Y) = \Gamma$ we may define a $\mathscr{B}$-measurable function $\tilde{\gamma}$ by the formula

$$\tilde{\gamma} = h(\gamma) \circ T. \tag{4}$$

We first prove the following result.

PROPOSITION. *The set $\Gamma(X) = \tilde{\Gamma} = \{\tilde{\gamma} \mid \gamma \in \Gamma\}$ is an Abelian group.*

Proof. Suppose $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \tilde{\Gamma}$. Then, there exist γ_1 and γ_2 such that $\tilde{\gamma}_1 = h(\gamma_1) \circ T$ and $\tilde{\gamma}_2 = h(\gamma_2) \circ T$. Hence

$$\begin{aligned}\tilde{\gamma}_1\tilde{\gamma}_2 &= (h(\gamma_1) \circ T)\,(h(\gamma_2) \circ T)\\ &= [h(\gamma_1)\, h(\gamma_2)] \circ T\\ &= h(\gamma_1\gamma_2) \circ T = h(\gamma_2\gamma_1) \circ T = \tilde{\gamma}_2\tilde{\gamma}_1 .\end{aligned}$$

Similarly,

$$(\tilde{\gamma})^{-1} = (h(\gamma) \circ T)^{-1} = h(\gamma^{-1}) \circ T.$$

Hence $\tilde{\Gamma}$ is a group.

For each $\tilde{\gamma} \in \tilde{\Gamma}$ we now define

$$\varphi_1(\tilde{\gamma}) = \int_Y h(\gamma)\, d\nu_2\,, \tag{5}$$

However, the choice of γ need not be unique, since $\tilde{\gamma} = h(\gamma) \circ T$. Hence, if $h(\gamma_1) \circ T = h(\gamma_2) \circ T$, we must show that

$$\int h(\gamma_1)\, d\nu_2 = \int h(\gamma_2)\, d\nu_2\,. \tag{6}$$

The above will obtain if we simply require that $T[X] \supseteq \mathrm{Supp}(\nu_2)$, i.e., the ν_2-outer measure of $T[X]$ is one. Otherwise, of course, (6) would not hold.

If we make the above assumption concerning $T[X]$, then the function $\varphi_1(\tilde{\gamma})$ is well defined. It is easy to show that φ_1 is positive-definite; we have

$$\begin{aligned}\sum_{i,j=1}^{n} z_i \bar{z}_j \varphi_1(\tilde{\gamma}_i(\tilde{\gamma}_j)^{-1}) &= \sum_{i,j=1}^{n} z_i \bar{z}_j \int h(\gamma_i)\, h(\gamma_j)\, d\nu_2 \\ &= \int \sum_{i,j=1}^{n} z_i \bar{z}_j h(\gamma_i)\, h(\gamma_j)\, d\nu_2 \\ &= \int \left| \sum_{i,j=1}^{n} z_i h(\gamma_i) \right|^2 d\nu_2 \geqslant 0\end{aligned}$$

for every choice of n, complex numbers $z_1, z_2, \ldots, z_n$, and elements $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n$ of $\tilde{\Gamma}$.

Thus, the only property of φ_1 that we have to establish is that $\varphi_1(\tilde{\gamma}) = 1$ if and only if $\tilde{\gamma}$ is the identity of $\tilde{\Gamma}$. However, if $\varphi_1(\tilde{\gamma}) = 1$, then $\int h(\gamma)\, d\nu = 1$; hence $h(\gamma) \equiv 1$ since (Γ, φ_2) is an algebraic measure system. Therefore, $\tilde{\gamma}$ is $1 \circ T = 1$, the identity of $\tilde{\Gamma}$. We can, therefore, conclude that $(\tilde{\Gamma}, \varphi_1)$ is an algebraic measure system independently of whether or not the measure ν_1 exists on $(X, \mathscr{B})$.

Let us now assume that ν_1 does exist and satisfies (3). It is important to observe that even if ν_1 exists it may not be unique, since (3) only holds for subsets of X of the form $T^{-1}(C)$, $C \in \mathscr{C}$. The class of all such sets is a sub-σ-algebra of $\mathscr{B}$. Hence in order that ν_1 be unique we require that $\mathscr{B}$ be precisely the σ-algebra of sets of the $T^{-1}(C)$ for every $C \in \mathscr{C}$.

We now state and prove the following result.

THEOREM 2. *Let* $T\colon (X, \mathscr{B}) \to (Y, \mathscr{C}, \nu_2)$ *be measurable with* $\nu_2{}^*(T[X]) = 1$, *and let (Γ, φ_2) be an algebraic model for $(Y, \mathscr{C}, \nu_2)$, with $h: \Gamma \to H(\nu_2)$. Then, the pair $(\tilde{\Gamma}, \varphi_1)$ defined by $\tilde{\Gamma} = \{\tilde{\gamma} = h(\gamma)T \mid \gamma \in \Gamma\}$ and $\varphi_1(\tilde{\gamma}) = \int h(\gamma)\, d\nu_2$ is an algebraic measure system. Furthermore, if the measure ν_1 on $(X, \mathscr{B})$ which satisfies the equation $\nu_1(T^{-1}(C)) = \nu_2(C)$ for all $C \in \mathscr{C}$ exists, and if $\mathscr{B}$ is the σ-algebra of sets of the form $T^{-1}(C)$ for every $C \in \mathscr{C}$, then $(\tilde{\Gamma}, \varphi_1)$ is an algebraic model for $(X, \mathscr{B}, \nu_1)$.*

Proof. It has already been shown that $(\tilde{\Gamma}, \varphi_1)$ is an algebraic measure system. Hence, it remains to be shown that if ν_1 exists, then $(\tilde{\Gamma}, \varphi_1)$ is an algebraic model for it. Since

$$\varphi_1(\tilde{\gamma}) = \int_Y h(\gamma)\, d\nu_2 = \int_X h(\gamma)\, d\nu_1(T^{-1})$$

$$= \int_X (h(\gamma) \circ T)\, d\nu_1 = \int_X \tilde{\gamma}\, d\nu_1 ,$$

we see that the first condition for $(\tilde{\Gamma}, \varphi_1)$ to be an algebraic model for ν_1 is satisfied.

We now show that the span of $\tilde{\Gamma}$ is dense in $L_2(\nu_1)$. Let $\tilde{f} \in L_2(\nu_1)$, and let $\{\tilde{g}_n\}$ be a sequence of simple functions converging to $\tilde{f}$ in $L_2(\nu_1)$-norm. Each $\tilde{g}_n$ must be of the form

$$\tilde{g}_n = \sum_{i=1}^{N_n} z_j{}^n \chi_{B_j{}^n}$$

for some N_n, and for some choice of $z_j{}^n \in Z$ and $B_j{}^n \in \mathscr{B}$, where the $B_j{}^n$ are pairwise disjoint. However, if $B_j{}^n \in \mathscr{B}$, then there exists a $C_j{}^n \in \mathscr{C}$ satisfying $T^{-1}(C_j{}^n) = B_j{}^n$. Hence, for each $\tilde{g}_n \in L_2(\nu_1)$ there is a measurable function g_n on Y satisfying the relation $\tilde{g}_n = g_n \circ T$. To show this, we choose

$$g_n = \sum_{j=1}^{N_n} z_j{}^n \chi_{C_j{}^n} .$$

Since $\nu_1(T^{-1}(C)) = \nu_2(C)$, we have

$$\int_X |\tilde{g}_n|^2\, d\nu_1 = \int_X |g_n \circ T|^2\, d\nu_1$$

$$= \int_X (|g_n|^2 \circ T)\, d\nu_1$$

$$= \int_Y |g_n|^2\, d\nu_1 T^{-1}$$

$$= \int_Y |g_n|^2\, d\nu_2 .$$

Therefore $g_n \in L_2(\nu_2)$, and since the sequence $\{\tilde{g}_n\}$ is Cauchy in $L_2(\nu_1)$ we see that $\{g_n\}$ is Cauchy in $L_2(\nu_2)$:

$$\int_X |\tilde{g}_n - \tilde{g}_m|^2 \, d\nu_1 = \int_X |g_n - g_m|^2 \circ T \, d\nu_1$$
$$= \int_Y |g_n - g_m|^2 \, d\nu_2 .$$

Hence, there exists an $f \in L_2(\nu_2)$ which is the limit of the g_n. Therefore

$$\int_X |\tilde{f} - f \circ T|^2 \, d\nu_1 = \lim_{n\to\infty} \int_X |\tilde{g}_n - f \circ T|^2 \, d\nu_1$$

by the dominated convergence theorem; and

$$\|\tilde{f} - f \circ T\|_2^2 = \lim_{n\to\infty} \int_X |g_n \circ T - f \circ T| \; d\nu_1$$
$$= \lim_{n\to\infty} \int_X |g_n - f|^2 \circ T \, d\nu_1$$
$$= \lim_{n\to\infty} \int_Y |g_n - f|^2 \, d\nu_2 = 0.$$

Therefore, $\tilde{f} = f \circ T$ in L_2. Since $f \in L_2(\nu_2)$ and (Γ, φ_2) is an algebraic model for $(Y, \mathscr{C}, \nu_2)$, given $\epsilon > 0$ there is a choice of $z_j \in Z$ and $\gamma_j \in \Gamma$, $j = 1, 2,..., n$ (for some n) such that

$$\epsilon > \int_Y \left| \sum_{j=1}^n z_j\gamma_j - f \right|^2 d\nu_2 = \int_Y \left| \sum_{j=1}^n z_j\gamma_j - f \right|^2 d\nu_1(T^{-1})$$
$$= \int_X \left(\left| \sum_{j=1}^n z_j\gamma_j - f \right|^2 \circ T \right) d\nu_1$$
$$= \int_X \left| \sum_{j=1}^n z_j\gamma_j \circ T - f \circ T \right|^2 d\nu_1$$
$$= \int_X \left| \sum_{j=1}^n z_j\tilde{\gamma}_j - \tilde{f} \right|^2 d\nu_1 ,$$

where we have put $\tilde{\gamma}_j = \gamma_j \circ T$. Therefore, every $\tilde{f} \in L_2(\nu_1)$ can be approximated by a linear combination of elements of $\tilde{\Gamma}$; and, under the assumption that ν_1 exists, $(\tilde{\Gamma}, \varphi_1)$ is an algebraic model for $(X, \mathscr{B}, \nu_1)$.

In view of the above, given a pair of probability measure spaces $(X, \mathscr{B}, \nu_1)$ and $(Y, \mathscr{C}, \nu_2)$ with T a measurable transformation from X to Y, and (i) $\mathscr{B}$ is minimal in the sense that it is generated by the preimage of $\mathscr{C}$, and (ii) $\nu_2{}^*(T[X]) = 1$, we have the property that for every algebraic model (Γ, φ_2) of

$(Y, \mathscr{C}, \nu_2)$ there exists an algebraic model $(\tilde{\Gamma}, \varphi_1)$ for $(X, \mathscr{B}, \nu_1)$ with $\tilde{\Gamma} = \{h(\gamma) \circ T \mid \gamma \in \Gamma\}$ and $\varphi_1(\tilde{\gamma}) = \varphi_2(\gamma)$.

If we now endow $(\tilde{\Gamma}, \varphi_1)$ with the discrete topology it will become a locally compact Abelian group. Let G denote the dual of $\tilde{\Gamma}$, and let $\mathscr{D}$ be the Borel algebra of G. Then (cf. [9, p. 19]) there exists a measure m on $(G, \mathscr{D})$ such that

$$\varphi_1(\tilde{\gamma}) = \int_G (\tilde{\gamma}, g)\, dm(g);$$

and, as was previously shown, $(\tilde{\Gamma}, \varphi_1)$ is an algebraic model for $(G, \mathscr{D}, m)$. Hence $(G, \mathscr{D}, m)$ and $(X, \mathscr{B}, \nu_1)$ are conjugate. In fact, following [12], we can say more.

THEOREM 3. *There is a measurable map λ: $X \to G$ and a conjugacy mapping V from $L_2(\nu_1)$ to $L_2(\nu_2)$ for which $v(f) \circ \lambda = f$ (ν_1 — a.s.) for each $f \in L_2(\nu_1)$.*

Proof. First, the conjugacy map λ is constructed in the natural way, using the fact that $\tilde{\Gamma}$ is a subset of both $L_2(\nu_1)$ and $L_2(m)$. For each $\tilde{\gamma} \in \tilde{\Gamma}$, regarded as a basis of $L_2(\nu_1)$, define the function $v(\tilde{\gamma}) = (\cdot, \tilde{\gamma})$ on G. Since $|v(\tilde{\gamma})| \leqslant 1$ it lies in $L_2(m)$ and in fact, since $(\tilde{\Gamma}, \varphi_1)$ is an algebraic model for $(G, \mathscr{D}, m)$, the set $v(\tilde{\Gamma})$ is a basis for $L_2(m)$. Hence, for any $L_2(\nu_1)$ sequence of linear combinations $f_n = \sum_{i=1}^{\infty} c_i^n \tilde{\gamma}_i^n$ approximating f we can define

$$v(f_n) = \sum_{i=1}^{\infty} c_i^n(\cdot, \tilde{\gamma}_i^n)$$

and set $v(f) =$ limit $v(f_n)$ in $L_2(m)$. Now define λ: $X \to G$ via $(\lambda(x), \tilde{\gamma}) = \tilde{\gamma}(x)$ for every $\tilde{\gamma} \in \tilde{\Gamma}$ and $x \in X$.

Now, to show $v(f) \circ \lambda = f(\nu_1 - \text{a.s.})$ we set, as before,

$$f_n = \sum_{i=1}^{\infty} c_i^n \tilde{\gamma}_i^n$$

in $L_2(\nu_1)$. Hence $v(f) =$ limit $v(f_n)$ in $L_2(m)$. Therefore, we have

$$\begin{aligned} v(f_n) \circ \lambda(x) &= \sum_{i=1}^{\infty} c_i^n(\cdot, \tilde{\gamma}_i^n) \circ \lambda(x) \\ &= \sum_{i=1}^{\infty} c_i^n(\lambda(x), \tilde{\gamma}_i^n) = \sum_{i=1}^{\infty} c_i^n \tilde{\gamma}_i^n(x). \end{aligned}$$

So, $v(f_n) \circ \lambda$ and f_n lie in the same $L_2(\nu_1)$ equivalence class; so that $v(f_n) \circ \lambda = f_n(\nu_1 - \text{a.s.})$. Since f is the L_2 limit of the f_n we see that likewise $v(f) \circ \lambda = f(\nu_1 - \text{a.s.})$.

The results of this section can be summarized by the following commutative diagram:

$$\begin{tikzcd}
(G, \mathscr{D}, m) \arrow[dr, leftrightsquigarrow] & (X, \mathscr{B}, \nu_1) \arrow[l, "\lambda"'] \arrow[r, "T"] \arrow[d, leftrightsquigarrow] & (Y, \mathscr{C}, \nu_2) \arrow[d, leftrightsquigarrow] \\
& (\tilde{\Gamma}, \phi_1) & (\Gamma, \phi_2) \arrow[l, "h(\cdot)\circ T"]
\end{tikzcd}$$

where $\leftrightsquigarrow$ denotes the algebraic model correspondence. If the measure ν_1 on $(X, \mathscr{B})$ does not exist the above diagram is not applicable; however, we are assured of the existence of $(G, \mathscr{D}, m)$ in any case.

References

1. A. T. Bharucha-Reid, "Random Integral Equations," Academic Press, New York, 1972.
2. M. J. Christensen and A. T. Bharucha-Reid, Algebraic models for Gaussian measures on Banach spaces, to appear.
3. N. Dinculeanu and C. Foiaş, Algebraic models for measures, *Illinois Math. J.* **12** (1968), 340–351.
4. M. P. Eršov, Extensions of measures and stochastic equations (Russian), *Teor. Verojatnost. i Primenen.* **14** (1974), 457–471.
5. I. I. Gikhman and A. V. Skorohod, On densities of probability measures in function spaces (Russian), *Uspehi Mat. Nauk* **21** (1966), 83–152.
6. I. I. Gikhman and A. V. Skorohod, "The Theory of Stochastic Processes I" (translated from the Russian), Springer–Verlag, New York, 1974.
7. P. R. Halmos, "Measure Theory," Van Nostrand, Princeton, N.J., 1950.
8. P.-S. Pun and A. T. Bharucha-Reid, Mesures associées à la solution de quelques équations intégrales de Fredholm aléatoires, *C. R. Acad. Sci. Paris Sér. A-B* **276** (1973), A699–A701.
9. W. Rudin, "Fourier Analysis on Groups," Interscience, New York, 1962.
10. A. D. Šatašvili, The transformation of measures in Hilbert space by means of linear differential equations (Russian), *Theory of Random Processes*, No. 2, pp. 113–120, Izdat. "Naukova Dumka," Kiev, 1974.
11. A. D. Šatašvili, Transformations of a Gaussian measure in Hilbert space that are generated by differential equations (Russian), *in* "Theory of Random Processes," No. 2, pp. 120–128, Izdat. "Naukova Dumka," Kiev, 1974.
12. B. M. Schreiber, T.-C. Sun, and A. T. Bharucha-Reid, Algebraic models for probability measures associated with stochastic processes, *Trans. Amer. Math. Soc.* **158** (1971), 93–105.
13. J. Yeh, "Stochastic Processes and the Wiener Integral," Dekker, New York, 1973.